Problem set 7: Economic Growth: The Solow Model

Problem 1 (HOMEWORK)

The production function is given by the following equation \( Y = F(K, N) = \left(\frac{K + N}{K}\right)^2 \), where \( Y, K, N \) denote, respectively, output and the capital and labor inputs. Verify whether this production function
a) exhibits constant returns to scale. Can the function be written in intensive form?
b) displays positive and diminishing marginal products of capital and labor. Under what conditions?
c) satisfies Inada conditions?

Problem 2 (HOMEWORK)

Is the following statement true or false? “It stems from the Solow growth model that size of an economy, measured by the level of GDP is a negative function of the rate of growth of population and capital depreciation rate. This is due to the fact steeper break-even investment line intersects the saving curve at lower level of capital stock.”

Problem 3

The aggregate wage bill in an economy is equal to 60 and output, which is produced according to the Cobb-Douglas production function, is equal to 100. The output growth rate was 10 percent and the growth rates of capital and labor were 10 percent and 5 percent respectively.

a) What was the overall productivity (TFP) growth rate for this economy? (Answer: 0.03)
b) Repeat (a) if the wage bill is 80 instead of 60. (Answer: 0.04)
c) Derive the expression for TFP growth in an economy that produces output according to the following production function:
\[ Y = (AK)^{\alpha} N^{\beta} T^{1-\alpha-\beta}, \]
where \( T \) denotes a fixed amount of arable land. (Answer: \((dA/dt)/A=(1/\alpha)[(dY/dt)/Y]-(\delta Y/K)-(\beta \alpha)[(dN/dt)/N]-(1-\alpha-\beta)/\alpha[(dT/dt)/T])\)

Problem 4

Poland’s entrance into the EU induced inflow of aid in the form a transfer of capital equipment. The Minister of Economy, the admirer of the Solow growth model, warned the people that the gift will result in higher value of per capita output only if people would start to save more. The Minister also said: “If the rate of saving would not increase, output will quickly return to the initial level. During this transition period the rate of economic growth will temporarily fall”. Was the Minister right?

Problem 5

Suppose we have two countries, AA and BB. They both have the same production function. Assume they start out with the same levels of capital, labour and technology and the capital-labour ratio is lower than the steady state level of capital per person. AA has a saving rate of 20 percent, whereas the saving rate in BB is equal to 25 percent. In both countries the growth
rate of population is 3 percent per year, depreciation rate is equal to 5 percent per year and the
pace of technical progress equals 3 percent per year. According to the Solow growth model
a) Which country, if either, currently has the higher growth rate of output per person? Why?
b) Which country, if either, will have a larger growth rate of output per person in the very
long run (i.e. in the steady state)? Why?
c) What is the growth rate of output in the steady state in both countries? (Give
numerical value).

Problem 6
An economy on the balanced growth path experiences a nasty natural disaster. A hurricane
destroys 50% of the economy’s population and 75% of its capital stock. Show the path of the
following variables (Note: your answer should consist of a graph of a variable or the log of a
variable on the vertical axis, and time on the horizontal):
a) Capital and production per worker \(k\) and \(y\)
b) Labor \(N\)
c) Capital stock \(K\)
d) Output \(Y\)

Problem 7
The economy is in the steady state. In a gloomy day the rate of depreciation of the capital
equipment increases from \(d_1\) to \(d_2\) (rust?) and one fourth of the labor force leaves the country
in pursuit of sunshine. The rate of population growth and the saving rate of people staying in
the country remains unchanged. The rate of technological progress is equal to zero. Show the
path of the following variables (Note: your answer should consist of a graph of a variable or
the log of a variable on the vertical axis, and time on the horizontal):
a) Capital and production per worker \(k\) and \(y\)
b) Labor \(N\)
c) Capital stock \(K\)
d) Output \(Y\)

Problem 8
Consider the Solow model with no technological progress. The production function in
intensive form is given by \(y = f(k) = \beta^\alpha k^{\alpha \beta}\) and the capital stock motion equation is
\(\dot{k} = s \beta (n + d) k\), where parameters have the usual meaning and \(\alpha \beta < 1\).
a) Calculate the steady-state value of per capita capital stock and production (Answer:
\(k^* = \frac{s \beta}{(n + d)} \frac{\beta}{1 - \alpha \beta}\))
b) Compute the golden-rule rate of savings (Answer: \(s_G = \alpha \beta\))
c) Sketch the evolution over time of the per capita consumption if a pension reform
increases the exogenous rate of saving from \(s_1 = 0.2\) to \(s_2 = 0.3\). Suppose \(\alpha = 0.8\) and
\(\beta = 0.4\).

Zadanie 9
The rate of growth of output \(Y\) is equal to 0.07, rate of population growth equals 0.01, and the
rate of growth of capital stock equals 0.03. Production function has a Cobb-Douglas
specification \(Y = K^\alpha (AN)^{1-\alpha}\), where \(K\) and \(N\) denote, respectively, capital and labor inputs
and \(\alpha = 0.5\). Using the growth accounting technique and the Solow growth model:
a) Calculate the rate of technological progress (Answer: \(g = 0.1\))
b) Calculate the level of real wage assuming that labor is paid its marginal product. What is the rate of growth of real wages if the economy is in the steady state? (Answer: wages grow at the rate of technological progress)

Problem 10
The production function in the intensive form reads as follows: \( \hat{y} = \alpha \hat{k}^\alpha \), where \( \hat{y} \) and \( \hat{k} \) denote, respectively, the level of output and capital per unit of effective labor \( AN \), and \( \alpha = \frac{1}{3} \). Assume that the saving rate \( s_1 = 0.3 \), rate of population growth \( n = -0.05 \), the depreciation rate \( d = 0.065 \), and the pace of technological progress \( g = 0.01 \). Using the Solow growth model:

a) Calculate the per unit of effective labor capital stock in the steady state (Answer: 8)

b) Calculate the per capita level of output if the level of technological advancement \( A = 30 \). (Answer: \( y = 20 \))

c) Write down the condition for maximization in the steady state of per unit of effective labor consumption and calculate the golden rule rate of saving. (Answer: \( s_G = \frac{1}{3} \))

d) Show the path of the log of per capita consumption before and after an increase of the saving rate to \( s_2 = 0.32 \).

Problem 11
The intensive form of the production function is written as: \( f(k) = \frac{k}{a(k + a)} \), where \( k \) is the capital stock per unit of effective labor, and \( a \) is a parameter, \( 0 < a < 1 \). The rate of technological progress, capital depreciation and population growth equal \( n, d \) and \( g \). Using the Solow growth model:

a) Verify whether the production function satisfies all necessary conditions.

b) What is the range of values of \( (n + d + g) \) for which the steady state in the Solow growth model can be reached? Explain and illustrate why a steady state may never be reached.

c) Suppose that \( (n + \delta + g) \) falls within the required range. Calculate the per unit of effective labor capital stock.

d) Suppose that \( (n + \delta + g) \) falls within the required range. Calculate the golden rule rate of saving.

Problem 12 (DIFFICULT)
Production function is given by \( Y = K^{\alpha} (AN)^{1-\alpha} \), where \( \alpha = \frac{1}{2} \). The saving rate = 0.4; rate of population growth = 0.01; the rate of technological progress = 0.02; capital depreciation rate = 0.04. The government decides to levy an income tax solely on wage income. The tax rate equals \( \tau = 0.25 \). Using the Solow growth model:

a) Write down the relationship between aggregate output \( Y \) and the sum of wage income and capital income (Answer: \( Y = wN + rK \))

b) Calculate total wage bill and total return to capital using the properties of the Cobb-Douglas production function. Plug your result in the expression obtained in (a). (Answer: \( Y = (1-\alpha)Y + \alpha Y \))

c) Using the results obtained in (a) and (b) write down the relationship between aggregate output \( Y \) and the sum of wage income and capital income after the imposition of income tax \( \tau \) on wage income. (Answer: \( Y_{disp} = (1-\alpha)(1-\tau)Y + \alpha Y \))
d) Calculate the value of saving, capital and production per unit of effective labor in the steady state after tax imposition. (Answer: \( S = Y_{disp} = s(1-(1-\alpha \tau)Y; \)
\[ \hat{k} = \left( \frac{s(1-(1-\alpha \tau))}{n+d+g} \right)^{1/\alpha} = 25 \]

e) Calculate the value of consumption after tax imposition. (Answer: \( \hat{c} = (1-s)(1-(1-\alpha \tau)) \left( \frac{s(1-(1-\alpha \tau))}{n+d+g} \right)^{1/\alpha} = 2,625 \)

Problem 13 (DIFFICULT)
In contrast to the original Solow growth model, suppose government expenditures (\( G \)) contribute to production because government spending makes private capital (\( K \)) and labor (\( N \)) more productive. Hence, the production function takes the following form:
\[ Y = AK^\alpha N^\beta G^\gamma, \]
where \( \alpha + \beta + \gamma = 1. \)
Assume for simplicity that the government runs a balanced budget which implies \( G = \tau Y \), where \( \tau \) is a fixed tax rate on output. Consumers save a constant fraction \( s \) of disposable income. The rate of population growth is equal to \( n \), the depreciation rate equals \( d \), and the rate of technological progress is given by \( z \).

a) Derive the steady-state condition for capital in units of effective labor, \( \hat{k} \), in terms of parameters and the level of tax rate, \( \tau \). Note that the units of effective labor cannot be expressed as \( AN \). You should instead use the definition of the form \( A^xN \) where \( x \) allows to write the production function in the following form:
\[ \frac{Y}{A^xN} = \left( \frac{K}{A^xN} \right)^\alpha \left( \frac{G}{A^xN} \right)^\gamma \equiv \hat{k}^x \hat{g}^\gamma. \]
Then make use of the relation \( G = \tau Y \). (Answer: \( \hat{k}^x = \left( \frac{s(1-\tau)\tau^{x-\gamma-1}}{n+d+x/\beta} \right)^{1/\beta} \))

b) How does \( g \) affect \( k \) in the steady state (HINT: Calculate the derivative of \( k \) with respect to \( \tau \) in the steady state and calculate the optimal tax rate). (Answer: \( \tau_{opt} = \gamma \))

c) What is the steady-state growth rate of \( Y \)? (Answer: \( n+x/\beta \))

Problem 14
The poverty trap concept is often used as an explicit motivation for foreign aid to help countries escape from underdevelopment. If either saving or productivity is low at low levels of development, investment will be low and countries will converge to an equilibrium with low capital and output per capita. If over some range of income levels saving rates and/or productivity increase sharply, then if countries can get to this point they might also be able to converge to an equilibrium with high capital and output per capita. In particular the saving rate is constant at some low rate until a threshold value of the capital stock is reached, and then it jumps to a constant higher rate, i.e.
\[ s(k) = \begin{cases} s_L & \text{for } k \leq \bar{k} \\ s_H & \text{for } k > \bar{k} \end{cases} \]
a) Use the basic Solow diagram to analyze the number of steady states in the economy characterized by the saving rate behavior described above. Use the same graph to provide the rationale for foreign aid.

b) The figures below display the relation between saving rates and the level of per capita capital stock in the group of all countries (Fig. 1) and low-income countries (Fig. 2). Explain whether the data presented in the figures corroborate the hypothesis of poverty trap.

Figure 1. Cross-country relation between saving and capital stocks per capita

![Figure 1](image1.png)

Figure 2. Relation between saving and capital stocks per capita in the group of low-income countries.

![Figure 2](image2.png)

c) Taking account of the actual pattern of saving rates displayed in Figures 1 and 2, use the basic Solow diagram to show all possible steady states equilibria in both groups of countries. Are all steady-states equilibria stable?